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Master's Thesis

Well-posedness of Dirichlet problem for an integro-differential equation with a small time derivative coefficient

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1 Introduction

Theory of differential equations with partial derivatives is a well-known mathematical field. It is not a surprise because it is frequently used in real-life problems in economics, biology, chemistry, physics, etc. These equations allow us to analyze processes, happening every day, through mathematical models.

The behaviour of a process, described by differential equations with partial derivatives, at every particular moment of time, usually, depend only on a state of the object at the current moment. However, the future development of a process can also depend on "previous moments". As usual in those situations, integral terms appear in corresponding models.

One of the widest set of equations that consider the history of a process is the set of integro-differential equations with partial derivatives of Volterra type [1],[2]. There are a lot of decent works, dedicated to researching those equations, for example [3].

Firstly, scientists need to prove a well-posedness of a problem which they are trying to solve. In other words, to show that there is an existence, uniqueness and continuous dependence on the initial data. For solving such problem in a case of differential equations with partial derivatives a method of a priori inequalities in negative norms can be used. It is well-described in works [4], [5], [6], [15]. This theory also helps with researching the existence of optimal control, build numerical methods, etc [7],[8],[9],[10],[11].

In our work we are dealing with an operator with a small time derivative coefficient.

$$\mathcal{L}_\varepsilon u = -\varepsilon\frac{\partial u}{\partial t} + u - \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right) d\tau, \qquad (1)$$

To work with it we can use the results of previous works.

In the work [12] a following equation was considered

$$u(x,t) - \int_0^t \sum_{i=1}^{n} K_i(t,\tau) \frac{\partial^2 u(x,\tau)}{\partial x_i^2} d\tau = f(x,t). \tag{2}$$

In that work the kernel $K(t,\tau)$ is the same for all the derivatives and it is considered to be non negative,

$$\int_0^T \int_0^T K(t,\tau) u(t) d\tau dt \geq 0, \forall u \in L_2([0,T]), \tag{3}$$

which is quite a limitation. In works [13],[14] the well-posedness of a Dirichlet problem was proved for the following equation.

$$u(x,t) - \sum_{i=1}^{n} \int_0^t K(t,\tau) u_{x_i x_i}(x,\tau) + (L_i(x,t,\tau) u(x,\tau)_{x_i} d\tau \tag{4}$$

$$- \int_0^t R(x,t,\tau) u(x,\tau) d\tau = f(x,t). \tag{5}$$

It was proven for a bigger class of kernels, which needed to be non negative only at the diagonal

$$K(t,t) \geq 0, \forall t \in [0,T]. \tag{6}$$

However, in these works there was a strong demanding condition of using only one kernel , the same for all the second derivatives.

Finally, in our work we are suggesting an approach, which allows us to prove a priori estimations in the case when different kernels correspond to different second derivatives. Moreover, we need kernels to be non negative only on diagonal. Also, in our work kernels can depend on spatial variable x.

2 General notation

Let us take a look at the following operator

$$\mathcal{L}_\varepsilon u = -\varepsilon \frac{\partial u}{\partial t} + u - \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(K_i(t,\tau,x) \frac{\partial u(x,\tau)}{\partial x_i} \right) d\tau, \tag{7}$$

And the corresponding conjugate operator

$$\mathcal{L}^*_\varepsilon u = \varepsilon \frac{\partial u}{\partial t} + u - \int_t^T \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(K_i(t,\tau,x) \frac{\partial u(x,\tau)}{\partial x_i} \right) d\tau, \tag{8}$$

where ε is some positive number. Then, we can show that for some small enough values of parameter ε, corresponding initial and boundary problems would be well-posed. We will consider cylindrical domain $Q = \Omega \times (0,T)$. We will consider $C^\infty_{BR}(\mathcal{Q})$ as a domain of the operator $\mathcal{L}_\varepsilon$. We shall assume that the kernels $K_i(t,\tau,x)$ meet the following conditions:

1. $K_i(t,\tau,x)$ are continuously-differentiable with respect to the variables t, τ, x (by K_i' we will denote the derivative of the function K_i with respect to the variable t);

2. There exists $\alpha > 0$, such that for all $i = 1, 2, \ldots, n$, $t \in [0,T]$, $x \in \Omega$ the following inequality holds

$$K_i(t,t,x) \geq \alpha.$$

Consider the parameter

$$\varepsilon \in \left(0; \frac{\alpha}{M\sqrt{2T}}\right). \tag{9}$$

As E, E_+, W, and W_+ we will denote completion of spaces C^∞_{BR}, C^∞_{BR*}, C^∞_{BR}, and C^∞_{BR*} with the respect of the following norms

$$\|u\|^2_E = \|u\|^2_{E_+} = \|u\|^2_{L_2(Q)} + \|u_t\|^2_{L_2(Q)} + \sum_{i=1}^n \|u_{x_i}\|^2_{L_2(Q)} = \tag{10}$$

$$= \int_Q u^2(x,t) + \sum_{i=1}^n u^2_{x_i}(x,t) dQ;$$

$$\|u\|_W^2 = \| \int_0^t u^2(x,t)d\tau \|_E^2 = \int_Q u^2(x,t)dQ+ \tag{11}$$
$$+ \int_Q \left(\int_0^t u^2(x,t)d\tau \right)^2 + \sum_{i=1}^n \left(\int_0^t u_{x_i}^(x,t)d\tau \right)^2 dQ;$$

$$\|u\|_{W_+}^2 = \| \int_0^T v^2(x,t)d\tau \|_E^2 = \int_Q v^2(x,t)dQ+ \tag{12}$$
$$+ \int_Q \left(\int_0^T v^2(x,t)d\tau \right)^2 + \sum_{i=1}^n \left(\int_0^T v_{x_i}^(x,t)d\tau \right)^2 dQ.$$

As previously, we will also consider negative spaces E^-, E_+^-, W^-, W_+^- and corresponding bilinear forms.

3 Supporting statements

First, we need to add notation, needed for the proofs of lemmas in this section.

Consider this linear operator:

$$\mathcal{L}u = u - \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(K_i(t,\tau,x) \frac{\partial u(x,\tau)}{\partial x_i} d\tau, \right) \tag{13}$$

here $u(x,t)$ is a function, which describes the system state, as previously, in the cylindrical domain $Q = \Omega \times (0,T)$, Ω - bounded domain in the n-dimensional space with a smooth boundary $\partial\Omega$.

Assume that kernels $K_i(t,\tau,x)$ meet the conditions that were set before.

Let M be the constant, that bounds K_i and K_i' from above in the whole domain Q.

The $\mathcal{L}$ operator domain is considered to be a space of infinitely differentiable functions, that satisfy the following initial and boundary conditions:

$$u|_{x\in\partial\Omega} = 0; \tag{14}$$

$$u|_{t=T} = 0. \tag{15}$$

The space of functions, that satisfy (14) and (15) will be denoted as $C^\infty_{BR}(Q)$. In the future, if the domain Q is considered, it will not be specified, but rather we will denote C^∞_{BR}.

Consider also the adjoint operator, defined as

$$\mathcal{L}^* u = u - \int_t^T \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(K_i(t,\tau,x) \frac{\partial u(x,\tau)}{\partial x_i} \right) d\tau. \tag{16}$$

The domain of the adjoint operator $\mathcal{L}^*$ will be a space of infinitely differentiable functions, that satisfy the following initial and boundary conditions

$$u|_{x\in\Omega} = 0; \tag{17}$$

$$u_{t=0} = 0. \tag{18}$$

This space will be denoted as $C^\infty_{BR^*}(Q)$.

We will denote with $H^1_0(Q)$ a completion of the space of smooth functions, that satisfy the initial and boundary conditions (14) - (15), with respect to the norm

$$||u||^2_{H^1_0} = ||u||^2_{L_2(Q)} + \sum_{i=1}^{n} ||u_{x_i}||^2_{L_2(Q)} = \tag{19}$$
$$= \int_Q u^2(x,t) + \sum_{i=1}^{n} u^2_{x_i}(x,t) dQ.$$

By S, S_+ we will denote the completion of the spaces $C^\infty_{BR}, C^\infty_{BR^*}$ with the respect to the norms

$$||u||^2_S = \left\| \int_0^t u(x,\tau) d\tau \right\|^2_{H^1_0} = \tag{20}$$
$$= \int_Q \left(\int_0^t u(x,\tau) d\tau \right)^2 + \sum_{i=1}^{n} \left(\int_0^t u_{x_i}(x,\tau) d\tau \right)^2 dQ;$$
$$||v||^2_{S_+} = \left\| \int_t^T v(x,\tau) d\tau \right\|^2_{H^1_0} = \tag{21}$$
$$= \int_Q \left(\int_t^T v(x,\tau) d\tau \right)^2 + \sum_{i=1}^{n} \left(\int_t^T v_{x_i}(x,\tau) d\tau \right)^2 dQ;$$

respectively.

By $H^{-1}_0(Q)$ let us denote the negative space, built from $H^1_0(Q)$. In detail, we can consider the space $L_2(Q)$ equipped with the norm

$$\|f\|_{H^{-1}_0} = \sup u \in H^1_0 \frac{(f,u)_{L_2(Q)}}{\|u\|_{H^1_0}}, \ f \in L_2(Q).$$

The completion of the space $\left(L_2(Q), ||f||_{H^{-1}_0}\right)$ is the space that was denoted by us with $H^{-1}_0(Q)$. Moreover, for all $f \in H^{-1}_0(Q)$

$$||f||_{H^{-1}_0} = \sup u \in H^1_0 \frac{(f,u)_{H^{-1}_0(Q) \times H^1_0(Q)}}{||u||_{H^1_0}},$$

where $(\cdot,\cdot)_{H^{-1}_0(Q) \times H^1_0(Q)}$ - extension of the bilinear form $(\cdot,\cdot)_{L_2(Q)}$ from $L_2(Q)$ to $H^{-1}_0(Q) \times H^1_0(Q)$ by continuity.

Lemma 3.1. *Let $F \in L_2(Q)$, $G \in L_2([0,T] \times Q)$, $\sup t \in [0,T] \times Q|G| \leq M$. Then, for an arbitrary constant $c > 0$ the following inequality holds:*

$$\int_Q e^{-ct} F(x,t) \int_0^t G(t,x,\tau) F(x,\tau) d\tau dQ \leq M \sqrt{\frac{T}{c}} \int_Q e^{-ct} F^2(x,t) dQ$$

Proof. For an arbitrary $x \in \Omega$, from the Cauchy inequality follows

$$\begin{aligned}
&\int_0^t G(t,x,\tau) F(x,\tau) d\tau = \int_0^t e^{c\tau/2} G(t,x,\tau) \cdot e^{-c\tau/2} F(x,t) d\tau \leq \\
&\leq \left(\int_0^t e^{c\tau} G^2(t,x,\tau) d\tau \right)^{1/2} \cdot \left(\int_0^t e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} \leq \\
&\leq M \cdot \left(\int_0^t e^{c\tau} d\tau \right)^{1/2} \cdot \left(\int_0^t e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} = \\
&= M \cdot \frac{(e^{ct} - 1)^{1/2}}{\sqrt{c}} \cdot \left(\int_0^t e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} \leq \\
&\leq \frac{M}{\sqrt{c}} \cdot e^{ct/2} \left(\int_0^t e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_Q e^{-ct} F(x,t) \int_0^t G(t,x,\tau) F(x,\tau) d\tau dQ \leq \\
&\leq \int_Q e^{-ct} F(x,t) \cdot \frac{M}{\sqrt{c}} \cdot e^{ct/2} \cdot \left(\int_0^t e^{-c\tau} F(x,\tau) d\tau \right)^{1/2} dQ = \\
&= \frac{M}{\sqrt{c}} \cdot \int_Q e^{-ct/2} F(x,t) \cdot \left(\int_0^t e^{-c\tau} F(x,\tau) d\tau \right)^{1/2} dQ \leq \\
&\leq \frac{M}{\sqrt{c}} \cdot \left(\int_Q e^{-ct} F^2(x,t) dQ \right)^{1/2} \cdot \left(\int_Q \int_0^t e^{-c\tau} F^2(x,\tau) dQ \right)^{1/2} \leq \\
&\leq \frac{M}{\sqrt{c}} \cdot \left(\int_Q e^{-ct} F^2(x,t) dQ \right)^{1/2} \cdot \left(\int_Q e^{-ct} F^2(x,t) dQ \right)^{1/2} = \\
&= M \sqrt{\frac{T}{c}} \cdot \left(\int_Q e^{-ct} F^2(x,t) dQ \right)^{1/2} \cdot \left(\int_Q e^{-ct} F^2(x,t) dQ \right)^{1/2} = \\
&= M \sqrt{\frac{T}{c}} \cdot \int_Q e^{-ct} F^2(x,t) dQ.
\end{aligned}$$

□

Lemma 3.2. *Let* $F \in L_2(Q)$, $G \in L_2([0,T] \times Q), \sup_{[0,T]\times Q} |G| \leq M$. *Then, for arbitrary constant* $c < 0$ *the following inequality holds*

$$\int_Q e^{-ct} F(x,t) \int_t^T G(t,x,\tau) F(x,\tau) d\tau dQ \leq M \frac{T}{|c|} \int_Q e^{-ct} F^2(x,t) dQ.$$

Proof. For arbitrary $x \in \Omega$, from the Cauchy inequality follows

$$\begin{aligned}
&\int_t^T G(t,x,\tau) F(x,\tau) d\tau = \int_t^T e^{c\tau/2} G(t,x,\tau) \cdot e^{-c\tau/2} F(x,t) d\tau \leq \\
&\leq \left(\int_t^T e^{c\tau} G^2(t,x,\tau) d\tau \right)^{1/2} \cdot \left(\int_t^T e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} \leq \\
&\leq M \cdot \left(\int_t^T e^{c\tau} d\tau \right)^{1/2} \cdot \left(\int_t^T e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} = \\
&= M \cdot \frac{|e^{cT} - e^{ct}|^{1/2}}{\sqrt{|c|}} \cdot \left(\int_t^T e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} = \\
&= M \cdot e^{ct/2} \cdot \frac{|e^{c(T-t)} - 1|^{1/2}}{\sqrt{|c|}} \cdot \left(\int_t^T e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2} \leq \\
&\leq \frac{M}{\sqrt{|c|}} \cdot e^{ct/2} \cdot \left(\int_t^T e^{-c\tau} F^2(x,\tau) d\tau \right)^{1/2}.
\end{aligned}$$

Then,

$$
\begin{aligned}
&\int_Q e^{-ct}F(x,t)\int_t^T G(t,x,\tau)F(x,\tau)d\tau dQ \le \\
&\le \int_Q e^{-ct}F(x,t)\cdot\frac{M}{\sqrt{|c|}}\cdot e^{ct/2}\cdot\left(\int_t^T e^{-c\tau}F(x,\tau)d\tau\right)^{1/2} dQ = \\
&= \frac{M}{\sqrt{|c|}}\cdot\int_Q e^{-ct/2}F(x,t)\cdot\left(\int_t^T e^{-c\tau}F(x,\tau)d\tau\right)^{1/2} dQ \le \\
&\le \frac{M}{\sqrt{|c|}}\cdot\left(\int_Q e^{-ct}F^2(x,t)dQ\right)^{1/2}\cdot\left(\int_Q\int_t^T e^{-c\tau}F^2(x,\tau)dQ\right)^{1/2} \le \\
&\le \frac{M}{\sqrt{|c|}}\cdot\left(\int_Q e^{-ct}F^2(x,t)dQ\right)^{1/2}\cdot\left(\int_Q\int_0^T e^{-c\tau}F^2(x,\tau)dQ\right)^{1/2} \le \\
&\le M\sqrt{\frac{T}{|c|}}\cdot\left(\int_Q e^{-ct}F^2(x,t)dQ\right)^{1/2}\cdot\left(\int_Q e^{-ct}F^2(x,t)dQ\right)^{1/2} = \\
&= M\sqrt{\frac{T}{|c|}}\cdot\int_Q e^{-ct}F^2(x,t)dQ.
\end{aligned}
$$

□

Lemma 3.3. *There exists such a constant $c_0 > 0$, so that for an arbitrary function $v \in H_0^1(Q)$ the following inequality holds*

$$\|\mathcal{L}^* v\|_{H_0^{-1}(Q)} \ge c_0\|v\|_{S_+}. \tag{22}$$

Proof. Let us begin with taking $v \in C^\infty_{BR^*}$. Consider the function u, that is defined by

$$\int_t^T v(x,s)ds = se^{-ct}u(x,t), \tag{23}$$

where the value of the constant $c > 0$ we will define later. The function u, defined in this way belongs to the space C^∞_{BR}. It is evident, that

$$v(x,t) = -\left(e^{-ct}u(x,t)\right)'_t. \tag{24}$$

Consider the value of the bilinear form

$$(\mathcal{L}u, v)_{L_2(Q)} = I_1 + I_2.$$

Where

$$I_1 = (u,v)_{L_2(Q)} = -\int_Q u(x,t)(e^{-ct}u(x,t))'_t = \tag{25}$$
$$= -\int_Q e^{ct} \cdot e^{-ct}u(x,t)(e^{-ct}u(x,t))'_t dQ =$$
$$= -\frac{1}{2}\int_\Omega \int_0^T e^{ct}\frac{d}{dt}(e^{-ct}u(x,t))^2 dt d\Omega.$$

Integrating by parts with the respect to the variable t, we get

$$I_1 = -\frac{1}{2}\int_\Omega \left(e^c t \left(e^{-ct}u(x,t)\right)^2 |_{t=0}^{t=T} + \int_0^T ce^{ct} \cdot ce^{-2ct}u^2(x,t)dt \right) d\Omega \geq \tag{26}$$
$$\geq \int_Q e^{-ct}u^2(x,t)dQ = \frac{c}{2}\|e^{-ct/2}u\|_{L_2(Q)}.$$

Now, consider the term

$$I_2 = \int_Q \int_0^t \sum_{i=1}^n (K_i(t,\tau,x)u(x,\tau)_{x_i})_{x_i} d\tau v(x,t) dQ. \tag{27}$$

Using the divergence theorem, we get

$$I_2 = -\int_Q \int_0^t \sum_{i=1}^n (K_i(t,\tau,x)u_{x_i}(x,\tau))_{x_i} d\tau v(x,t) dQ = \tag{28}$$
$$= \int_Q \int_0^t \sum_{i=1}^n K_i(t,\tau,x)u_{x_i}(x,\tau) d\tau v_{x_i}(x,t) dQ =$$
$$= \int_\Omega \int_0^T \int_0^t \sum_{i=1}^n K_i(t,\tau,x)u_{x_i}(x,t) d\tau v_{x_i}(x,t) dt d\Omega.$$

Now, integrating by parts and recalling (23)

$$\int_0^T \int_0^t K_i(t,\tau,x) u_{x_i}(x,\tau) d\tau v_{x_i}(x,t) dt = \tag{29}$$

$$= \int_0^t K_i(t,\tau,x) u_{x_i}(x,t) d\tau \cdot \int_t^T -v_{x_i}(x,s) ds|_{t=0}^{t=T} -$$

$$- \int_0^T \left(\int_0^t K_i(t,\tau,x) u_{x_i}(x,\tau) d\tau \right)_t' \cdot \int_t^T -v_{x_i}(x,s) ds dt =$$

$$= \int_0^T \left(\int_0^t K_i(t,\tau,x) u_{x_i}(x,\tau) d\tau \right)_t' \cdot \int_t^T v_{x_i}(x,s) ds dt =$$

$$= \int_0^T \left(\int_0^t K_i(t,\tau,x) u_{x_i}(x,\tau) d\tau \right)_t' \cdot e^{-ct} u_{x_i}(x,t) dt =$$

$$= \int_0^T \left(\int_0^t K_i'(t,\tau,x) u_{x_i}(x,\tau) d\tau + K_i(t,t,x) u_{x_i}(x,t) \right) \cdot e^{-ct} u_{x_i}(x,t) dt.$$

Hence,

$$I_2 = \sum_{i=1}^n \int_Q \left(\int_0^t K_i'(t,\tau,x) u_{x_i}(x,\tau) d\tau + K_i(t,t,x) u_{x_i}(x,t) \right) \cdot e^{-ct} u_{x_i}(x,t) dQ = \tag{30}$$

$$= \sum_{i=1}^n \int_Q \int_0^t K_i'(t,\tau,x) u_{x_i}(x,\tau) d\tau e^{-ct} u_{x_i}(x,t) dQ +$$

$$+ \sum_{i=1}^n \int_Q K_i(t,t,x) e^{-ct} u_{x_i}^2(x,t) dQ.$$

Now, applying lemma 3.1, we get

$$\int_Q e^{-ct} u_{x_i}(x,t) \int_0^t K_i'(t,\tau,x) u_{x_i}(x,\tau) d\tau dQ \leq M \sqrt{\frac{T}{c}} \int_Q e^{-ct} u^2(x,t) dQ.$$

Using this in the (30), we get

$$I_2 \geq \sum_{i=1}^{n} \int_Q K_i(t,t,x) e^{-ct} u_{x_i}^2(x,t) dQ - \tag{31}$$
$$- \sum_{i=1}^{n} \left| \int_Q \int_0^t K_i'(t,\tau,x) u_{x_i}(x,\tau) d\tau e^{-ct} u_{x_i}(x,t) dQ \right| \geq$$
$$\geq \sum_{i=1}^{n} \alpha \int_Q e^{-ct} u_{x_i}^2(x,t) dQ - \sum_{i=1}^{n} M \sqrt{\frac{T}{c}} \int_Q e^{-ct} u_{x_i}^2(x,t) dQ \geq$$
$$\geq \sum_{i=1}^{n} \left(\alpha - M \sqrt{\frac{T}{c}} \right) \| e^{-ct/2} u_{x_i} \|_{L_2(Q)}^2 .$$

By adding inequalities (26) and (31), and considering their respective estimations, we get

$$(\mathcal{L}u, v)_{L_2(Q)} = I_1 + I_2 \geq \frac{c}{2} \| e^{-ct/2} u \|_{L_2(Q)}^2 + \tag{32}$$
$$+ \sum_{i=1}^{n} \left(\alpha - M \sqrt{\frac{T}{c}} \right) \| e^{-ct/2} u_{x_i} \|_{L_2(Q)}^2 \geq$$
$$\geq \frac{c}{2} \| e^{-ct/2} u \|_{L_2(Q)}^2 \geq c_1 \| u \|_{H_0^1(Q)}^2 \geq c_0 \| u \|_{H_0^1(Q)} \| v \|_{S_+} .$$

Estimating the value of the bilinear form with the Schwarz inequality, we get

$$\| u \|_{H_0^1(Q)} \cdot \| \mathcal{L}^* v \|_{H_0^{-1}} \geq (u, \mathcal{L}^* v)_{H_0^1(Q) \times H_0^{-1}(Q)} \geq c_0 \| v \|_{S_+} \| u \|_{H_0^1(Q)} . \tag{33}$$

Dividing by $\| u \|_{H_0^1(Q)}$, we get the lemma statement $\forall v \in C_{BR^*}^{\infty}$. For an arbitrary $v \in H_0^1(Q)$, by choosing the sequence $v_i \in C_{BR^*}^{\infty}$, that goes to v and going to the limit, we get the lemma statement for all $v \in H_0^1(Q)$. □

Lemma 3.4. *There exists a constant $c_0 > 0$, such that for an arbitrary function $u \in H_0^1(Q)$ the following inequality holds*

$$\| \mathcal{L} u \|_{H_0^{-1}(Q)} \geq c_0 \| u \|_S . \tag{34}$$

Proof. Let $v \in C_{BR^*}^{\infty}$. Consider a function u, defined as

$$\int_0^t u(x,s) ds = e^{-ct} v(x,t), \tag{35}$$

where the value of the constant $c>0$ will be defined later. Note, that function u, defined in this way belongs to the space C^*_{BR}. It is evident, that

$$u(x,t)=\left(e^{-ct}v(x,t)\right)'_t. \tag{36}$$

Consider the value of the bilinear form

$$(\mathcal{L}^*v,u)_{L_2(Q)}=I_1+I_2,$$

Then,

$$\begin{aligned} I_1=(u,v)_{L_2(Q)}&=\int_Q v(x,t)\left(e^{-ct}v(x,t)\right)'_t= \\ &=\int_Q e^{ct}\cdot e^{-ct}v(x,t)\left(e^{-ct}v(x,t)\right)'_t dQ= \\ &=\frac{1}{2}\int_\Omega\int_0^T e^{ct}\frac{d}{dt}\left(e^{-ct}v(x,t)\right)^2 dtd\Omega. \end{aligned} \tag{37}$$

Integrating bu parts with the respect to the variable t, we get

$$\begin{aligned} I_1&=\frac{1}{2}\int_\Omega\left(e^{ct}(e^{-ct}v(x,t))^2|_{t=0}^{t=T}-\int_0^T ce^{ct}\cdot e^{-2ct}v^2(x,t)dt\right)d\Omega \\ &\geq-\frac{c}{2}\int_Q e^{-ct}v^2(x,t)dQ=-\frac{c}{2}\|e^{-ct/2}v\|_{L_2(Q)}. \end{aligned} \tag{38}$$

Now, consider the term

$$I_2=-\int_Q\int_t^T\sum_{i=1}^n\left(K_i(\tau,t,x)v(x,\tau)_{x_i}\right)_{x_i}d\tau u(x,t)dQ. \tag{39}$$

Using the divergence theorem, we obtain

$$\begin{aligned} I_2&=-\int_Q\int_t^T\sum_{i=1}^n\left(K_i(\tau,t,x)v_{x_i}(x,\tau)\right)_{x_i}d\tau u(x,t)dQ= \\ &=\int_Q\int_t^T\sum_{i=1}^n K_i(\tau,t,x)v_{x_i}(x,\tau)d\tau u_{x_i}(x,\tau)dQ= \\ &=\int_\Omega\int_0^T\int_t^T\sum_{i=1}^n K_i(\tau,t,x)v_{x_i}(x,\tau)d\tau u_{x_i}(x,t)dtd\Omega. \end{aligned} \tag{40}$$

Now, integrating by parts, and recalling (35), we get

$$\int_0^T \int_t^T K_i(\tau,t,x) v_{x_i}(x,\tau) d\tau u_{x_i}(x,t) dt = \tag{41}$$

$$= \int_t^T K_i(\tau,t,x) v_{x_i}(x,\tau) d\tau \cdot \int_0^t u_{x_i}(x,s) ds \Big|_{t=0}^{t=T} -$$

$$- \int_0^T \left(\int_t^T K_i(\tau,t,x) v_{x_i}(x,\tau) d\tau \right)_t' \cdot \int_0^t u_{x_i}(x,s) ds dt =$$

$$= \int_0^T \left(\int_t^T K_i(\tau,t,x) v_{x_i}(x,\tau) d\tau \right)_t' \cdot \int_0^t u_{x_i}(x,s) ds dt =$$

$$= \int_0^t \left(\int_t^T K_i(\tau,t,x) v_{x_i}(x,\tau) d\tau \right)_t' \cdot e^{-ct} v_{x_i}(x,t) dt =$$

$$= \int_0^T \left(\int_t^T K_i'(\tau,t,x) v_{x_i}(x,\tau) d\tau - K_i(t,t,x) v_{x_i}(x,t) \right) \cdot e^{-ct} v_{x_i}(x,t) dt.$$

Hence, we get

$$I_2 = -\sum_{i=1}^n \int_Q \left(\int_0^t K_i'(\tau,t,x) v_{x_i}(x,\tau) d\tau + K_i(t,t,x) v_{x_i}(x,t) \right) \cdot e^{-ct} v_{x_i}(x,t) dQ =$$

$$\tag{42}$$

$$= \sum_{i=1}^n \int_Q \int_0^t K_i'(\tau,t,x) v_{x_i}(x,\tau) d\tau e^{-ct} v_{x_i}(x,t) dQ +$$

$$+ \sum_{i=1}^n \int_Q K_i(t,t,x) e^{-ct} v_{x_i}^2(x,t) dQ.$$

From the lemma 3.2, we get

$$\int_Q e^{-ct} v_{x_i}(x,t) \int_t^T K_i'(\tau,t,x) v_{x_i}(x,\tau) d\tau dQ \le M \sqrt{\frac{T}{c}} \int_Q e^{-ct} v^2(x,t) dQ.$$

Utilizing this in the equation (42), we obtain

$$I_2 \geq \sum_{i=1}^{n} \int_Q K_i(t,t,x) e^{-ct} v_{x_i}^2 dQ - \tag{43}$$

$$- \sum_{i=1}^{n} \left| \int_Q \int_0^t K_i'(\tau, t, x,) u_{x_i}(x,\tau) d\tau e^{-ct} v_{x_i}(x,t) dQ \right| \geq$$

$$\geq \sum_{i=1}^{n} \alpha \int_Q e^{-ct} v_{x_i}^2(x,t) dQ - \sum_{i=1}^{n} M \sqrt{\frac{T}{c}} \int_Q e^{-ct} v_{x_i}^2 dQ \geq$$

$$\geq \sum_{i=1}^{} \left(\alpha - M\sqrt{\frac{T}{|c|}} \right) \|e^{-ct/2} v_{x_i}\|_{L_2(Q)}^2. \tag{44}$$

By adding inequalities (38) and (43), we obtain the estimation

$$(\mathcal{L}u, v)_{L_2(Q)} = I_1 + I_2 \geq -\frac{c}{2} \|e^{-ct/2} v\|_{L_2(Q)}^2 + \tag{45}$$

$$+ \sum_{i=1}^{n} \left(\alpha - M \frac{T}{|c|} \right) \|e^{-ct/2} v_{x_i}\|_{L_2(Q)}^2 \geq$$

$$\geq -\frac{c}{2} \|e^{-ct/2} v\|_{L_2(Q)}^2 \geq c_1 \|v\|_{H_0^1(Q)}^2 \geq c_0 \|v\|_{H_0^1(Q)} \|u\|_S. \tag{46}$$

By estimating the value of the bilinear form according to the Schwarz inequality, we get

$$\|v\|_{H_0^1(Q)} \cdot \|\mathcal{L}u\|_{H_0^{-1}} \geq (\mathcal{L}u, v)_{H_0^1(Q) \times H_0^{-1}(Q)} \geq c_0 \|u\|_S \|v\|_{H_0^1(Q)}. \tag{47}$$

Eliminating $\|u\|_{H_0^1(Q)}$, we get the lemma statement for all $v \in C_{BR^*}^{\infty}$. For an arbitrary $v \in H_0^1(Q)$, by choosing the sequence $v_i \in C_{BR}^{\infty}$, such that it has for its limit v, and using the going to the limit, we get the lemma statement for all $v \in H_0^1(Q)$. □

4 A priori estimations

Lemma 4.1. *Let there be some constant $c_1 > 0$, such that for an arbitrary function $v \in C^\infty_{BR}$ the following inequality holds*

$$\|\mathcal{L}_\varepsilon u\|_{E^-_+} \leq c_1 \|u\|_E. \tag{48}$$

Proof. For some arbitrary $u \in C^\infty_{BR}, v \in C^\infty_{BR^*}$ we will take a look at the values of a bilinear form $(\mathcal{L}_\varepsilon u, v)_{L_2(Q)}$. Now we have:

$$\begin{aligned}(\mathcal{L}_\varepsilon u, v)_{L_2(Q)} &= (-\varepsilon \frac{\partial u}{\partial t}, v)_{L_2(Q)} + (u, v)_{L_2(Q)} - \\ &- \sum_{i=1}^n \left(\int_0^t \frac{\partial}{\partial x_i} \left(K_i(t, \tau, x) \frac{\partial u(x, \tau)}{\partial x_i} \right) d\tau, u \right)_{L_2(Q)}.\end{aligned} \tag{49}$$

Applying the Cauchy inequality we will receive

$$\begin{aligned}|(\mathcal{L}_\varepsilon u, v)_{L_2(Q)}| &\leq |(-\varepsilon \frac{\partial u}{\partial t}, v)_{L_2(Q)}| + |(u, v)_{L_2(Q)}| - \\ - \sum_{i=1}^n &\left| \left(\int_0^t \frac{\partial}{\partial x_i} \left(K_i(t, \tau, x) \frac{\partial u(x, \tau)}{\partial x_i} \right) d\tau, u \right)_{L_2(Q)} \right| \leq \\ &\leq \varepsilon \|u'_t\|_{L_2(Q)} \|v\|_{L_2(Q)} + \|u\|_{L_2(Q)} \|v\|_{L_2(Q)} - \\ - \sum_{i=1}^n &\left| \left(\int_0^t \frac{\partial}{\partial x_i} \left(K_i(t, \tau, x) \frac{\partial u(x, \tau)}{\partial x_i} \right) d\tau, u \right)_{L_2(Q)} \right|.\end{aligned} \tag{50}$$

Let us take a look at the sum from the previous formula. We will apply Gaussian formula and use an integration by parts with the respect to t we will receive the following:

$$I_3 = \left| \int_Q \int_0^t K_i(t,\tau,x) u_{x_i}(x,\tau) d\tau \frac{d}{dt} \left(\int_t^T v_{x_i}(x,s) ds \right) dQ \right| \leq \tag{51}$$

$$\leq \left| \int_Q \int_0^t K_i'(t,\tau,x) u_{x_i}(x,\tau) d\tau \int_t^T v_{x_i}(x,s) ds dQ \right| +$$

$$+ \left| \int_Q K_i(t,t,x) u_{x_i}(x,t) \int_t^T v_{x_i}(x,s) ds dQ \right| \leq$$

$$\leq M \left| \int_Q \int_0^t |u_{x_i}(x,\tau)| d\tau \int_t^T |v_{x_i}(x,s)| ds dQ \right| +$$

$$+ M \left| \int_Q |u_{x_i}(x,\tau)| \int_t^T |v_{x_i}(x,s)| ds dQ \right|.$$

Using the Cauchy and Friedrichs' inequalities we will get

$$I_3 \leq M \left\| \int_0^t |u_{x_i}(x,\tau)| d\tau \right\|_{L_2(Q)} \cdot \left\| \int_t^T |v_{x_i}(x,s)| ds \right\|_{L_2(Q)} + \tag{52}$$

$$+ M \|u_{x_i}\|_{L_2(Q)} \cdot \left\| \int_t^T |v_{x_i}(x,s)| ds \right\|_{L_2(Q)} \leq$$

$$\leq c_1 M \left(\|u_{x_i}\|_{L_2(Q)} \cdot \|v_{x_i}\|_{L_2(Q)} + \|v_{x_i}\|_{L_2(Q)} \cdot \|u_{x_i}\|_{L_2(Q)} \right).$$

Thus,

$$I_3 \leq c_2 \|u\|_E \|v\|_{E_+}. \tag{53}$$

So,

$$|(\mathcal{L}_\varepsilon u, v)_{L_2(Q)}| \leq \varepsilon \|u_t'\|_{L_2(Q)} \|v\|_{L_2(Q)} + \|u\|_{L_2(Q)} \|v\|_{L_2(Q)} \tag{54}$$

$$+ c_2 \|u\|_E \|v\|_{E_+} \leq c_4 \|u\|_E \|v\|_{E_+}.$$

Next, we can divide by $\|v\|_{E_+}$ and taking the supremum over all $y \in C_{BR}^\infty$ we will receive exactly the statement of this lemma. □

Using the analogical ideas the following lemma can be easily proven.

Lemma 4.2. *There exists a constant $c_1 > 0$, such that for an arbitrary function $v \in C_{BR}^\infty$ the following inequality holds:*

$$c \|v\|_{E_+} \geq \|\mathcal{L}^* v\|_{E^-}.$$

Lemma 4.3. *There exists a constant $c_0 > 0$, such that for an arbitrary function $u \in E$ the following inequality holds:*

$$\|\mathcal{L}^* v\|_{E^-} \geq c\|v\|_{W_+}. \tag{55}$$

Proof. Let $u \in C^*_{BR}$. Analogously to the proof of lemma 3.3 we shall use a test function, defined by the equation (23). Consider the corresponding value of the bilinear form

$$(\mathcal{L}_\varepsilon u, v) = (-\varepsilon u_t, v) + (\mathcal{L}u, v). \tag{56}$$

Using the results from the proof of lemma 3.3, we can get an estimate (32):

$$(\mathcal{L}u, v)_{L_2(Q)} \geq \frac{c}{2}\|e^{-ct/2}u\|^2_{L_2(Q)} + \sum_{i=1}^{n}\left(\alpha - M\frac{T}{c}\right)\|e^{-ct/2}u_{x_i}\|^2_{L_2(Q)}.$$

Considering the first term of the equation (56):

$$\begin{aligned}(-\varepsilon u_t, v)_{L_2(Q)} &= -\varepsilon\int_Q u_t(x,t)(-e^{-ct}u(x,t))'_t dQ = \\ &= \varepsilon\int_Q u_t(x,t)(e^{-ct}u(x,t))'_t dQ = \\ &= \varepsilon\int_Q u_t(x,t)(-ce^{-ct}u(x,t) + e^{-ct}u_t(x,t))dQ = \\ &= \varepsilon\|e^{-ct/2}u_t\|^2_{L_2(Q)} - c\varepsilon\int_Q e^{-ct}u_t(x,t)u(x,t)dQ.\end{aligned} \tag{57}$$

Using the Cauchy inequality for the last term, we obtain

$$(-\varepsilon u_t, v)_{L_2(Q)} \geq \varepsilon\|e^{-ct/2}u_t\|^2_{L_2(Q)} - c\varepsilon\|e^{-ct/2}u_t\|_{L_2(Q)}\|e^{-ct/2}u\|_{L_2(Q)}. \tag{58}$$

From inequalities (32) and (58), we finally get

$$\begin{aligned}(\mathcal{L}_\varepsilon u, v)_{L_2(Q)} &\geq \varepsilon\|e^{-ct/2}u_t\|^2_{L_2(Q)} - c\varepsilon\|e^{-ct/2}u_t\|L_2(Q)\|e^{-ct/2}u\|_{L_2(Q)} + \\ &+ \frac{c}{2}\|e^{-ct/2}u\|^2_{L_2(Q)} + \sum_{i=1}^{n}\left(\alpha - M\frac{T}{c}\right)\|e^{-ct/2}u_{x_i}\|^2_{L_2(Q)} \geq \\ &\geq \frac{\varepsilon}{2}\|e^{-ct/2}u_t\|^2_{L_2(Q)} - c\varepsilon\|e^{-ct/2}u_t\|_{L_2(Q)}\|e^{-ct/2}u\|_{L_2(Q)} + \\ &+ \frac{c}{4}\|e^{-ct/2}u\|^2_{L_2(Q)} + c_1\|e^{-ct/2}u\|^2_E,\end{aligned} \tag{59}$$

where $c_1 < \min\left\{\frac{\varepsilon}{2}, \frac{c}{4}, \alpha - M\sqrt{\frac{T}{c}}\right\}$. Therefore, as

$$\varepsilon < \frac{\alpha}{M\sqrt{2T}},$$

we can choose $c > 0$, such that

$$\varepsilon \leq \frac{1}{2c} \leq \frac{\alpha}{M\sqrt{2T}}.$$

Then,

$$\frac{\varepsilon}{2}A^2 + \frac{c}{4}B^2 \geq 2\sqrt{\frac{\varepsilon c}{8}}AB \geq \frac{1}{\sqrt{2}}\frac{\varepsilon c}{\sqrt{\varepsilon c}}AB \geq c\varepsilon AB. \tag{60}$$

Letting $A = \|e^{-ct/2}u_t\|^2_{L_2(Q)}$ and $B = \|e^{-ct/2}u\|^2_{L_2(Q)}$, from the inequality (59), we obtain

$$(\mathcal{L}_\varepsilon u, v)_{L_2(Q)} \geq c_1\|u\|^2_E = c_1\|u\|_E\|v\|_{W_+}. \tag{61}$$

Hence,

$$\|u\|_E\|\mathcal{L}^*_\varepsilon v\|_{E^-} \geq (u, \mathcal{L}^*_\varepsilon v)_{L_2(Q)} \geq c_1\|u\|^2_E = c_1\|u\|_E\|v\|_{W_+}. \tag{62}$$

and

$$\|\mathcal{L}^*_\varepsilon v\|_{E^-} \geq c_1\|v\|_{W_+}, \tag{63}$$

for an arbitrary $u \in C^\infty_{BR}$. For other $u \in E$ the lemma statement is being made with going to the limit. □

Lemma 4.4. *For an arbitrary function $v \in E^-_+$ there exists some constant $c_0 > 0$, such that the following inequality holds:*

$$\|\mathcal{L}_\varepsilon u\| \geq c_1\|u\|_W. \tag{64}$$

Proof. Let there be $v \in C^\infty_{BR^*}$. As in a proof of Lemma 3.4 we can use a test function given by the equation (35). Corresponding bilenear form would look like this:

$$(\mathcal{L}_\varepsilon u, v) = (u, \mathcal{L}^*_\varepsilon v) = (\varepsilon v_t, u) + (u, \mathcal{L}^* v). \tag{65}$$

Thus, using the results, gained in Lemma 3.4 we can claim the following set of inequalities:

$$(\mathcal{L}^* v, u)_{L_2(Q)} \geq -\frac{c}{2}\|e^{-ct/2}v\|^2_{L_2(Q)} + \sum_{i=1}^{n}\left(\alpha - M\sqrt{\frac{T}{|c|}}\right)\|e^{-ct/2}v_{x_i}\|^2_{L_2(Q)}.$$

The first term of (65) would give us the following:

$$\begin{aligned}(\varepsilon v_t, u)_{L_2(Q)} &= \varepsilon \int_Q v_t(x,t)(e^{-ct}v(x,t))'_t dQ = \qquad (66)\\ &= \varepsilon \int_Q v_t(x,t)(-ce^{-ct}v(x,t)) + e^{-ct}v_t(x,t))dQ = \\ &= \|e^{-ct/2v_t}\|^2_{L_2(Q)} - c\varepsilon \int_Q e^{-ct}v_t(x,t)v(x,t)dQ.\end{aligned}$$

Now, applying the Cauchy inequality to the last term we can get:

$$\begin{aligned}(\varepsilon v_t, u)_{L_2(Q)} &\geq \varepsilon\|e^{-ct/2v_t}\|^2_{L_2(Q)} + c\left|\varepsilon \int_Q e^{-ct}v_t(x,t)v(x,t)dQ\right| \geq \qquad (67)\\ &\geq \varepsilon\|e^{-ct/2v_t}\|^2_{L_2(Q)} + c\varepsilon\|e^{-ct/2v}\|_{L_2(Q)}\|e^{ct/2}v\|_{L_2(Q)}.\end{aligned}$$

Summing up it with (45) we receive

$$\begin{aligned}(\mathcal{L}^* v, u)_{L_2(Q)} &\geq \varepsilon\|e^{-ct/2v_t}\|^2_{L_2(Q)} + c\varepsilon\|e^{-ct/2v_t}\|^2_{L_2(Q)}\|e^{-ct/2v}\|^2_{L_2(Q)} - \qquad (68)\\ &- c/2\|e^{-ct/2v}\|^2_{L_2(Q)} + \sum_{i=1}^{n}\left(\alpha - M\sqrt{\frac{T}{|c|}}\right)\|e^{-ct/2v_{x_i}}\|^2_{L_2(Q)} \geq \qquad (69)\\ &\geq \frac{\varepsilon}{2}\|e^{-ct/2v_t}\|^2_{L_2(Q)} + c\varepsilon\|e^{-ct/2v_t}\|^2_{L_2(Q)}\|e^{-ct/2v}\|^2_{L_2(Q)} - \qquad (70)\\ &- \frac{c}{4}\|e^{-ct/2v}\|^2_{L_2(Q)} + c_1\|e^{-ct/2v}\|^2_E, \qquad (71)\end{aligned}$$

where $c_1 < \min\left\{\frac{\varepsilon}{2}, \frac{|c|}{4}, \alpha - M\sqrt{\frac{T}{|c|}}\right\}$. Because of

$$\varepsilon \leq \frac{\alpha}{M\sqrt{2T}}, \qquad (72)$$

we can choose $c \leq 0$ such that:

$$\varepsilon \leq \frac{1}{2|c|} < \frac{\alpha}{M\sqrt{2T}}. \tag{73}$$

Then,

$$\frac{\varepsilon}{2}A^2 + \frac{|c|}{4}B^2 \geq 2\sqrt{\frac{\varepsilon|c|}{8}}AB \geq \frac{\varepsilon|c|}{\sqrt{2\varepsilon|c|}}AB \geq |c|\varepsilon AB \tag{74}$$

If we choose $A = \|e^{-ct/2}v_t\|^2_{L_2(Q)}$ and $B = \|e^{-ct/2}v\|^2_{L_2(Q)}$, we can get from the (68) the following:

$$(\mathcal{L}^*_\varepsilon v, u)_{L_2(Q)} \geq c_1\|v\|^2_{E_+} = c_1\|v\|_{E_+}\|u\|_W. \tag{75}$$

Thus,

$$\|\mathcal{L}_\varepsilon u\|_{E^-}\|v\|_{E_+} \geq (\mathcal{L}_\varepsilon v, u)_{L_2(Q)} \geq c_1\|v\|^2_{E_+} = c_1\|v\|_{E_+}\|u\|_W, \tag{76}$$

which can give us the following result:

$$\|\mathcal{L}_\varepsilon u\|_{E^-_+} \geq c_1\|u\|_W, \tag{77}$$

for some arbitrary $v \in C^\infty_{BR^*}$. For other $v \in E_+$ we can receive the lemma's statement by going to the limit. □

To sum it all up, we can formulate a following theorem for operators $\mathcal{L}_\varepsilon, \mathcal{L}^*_\varepsilon$:

Theorem 4.1. *There exist such constants $c_0, c_1 > 0$, such that for some arbitrary $u \in E, v \in E_+$ the following inequalities hold*

$$\begin{cases} c_0\|v\|_W \leq \|\mathcal{L}_\varepsilon u\|_{E^-_+} \leq c_1\|u\|_E, \\ c_0\|v\|_{W_+} \leq \|\mathcal{L}^*_\varepsilon v\|_{E^-} \leq c_1\|u\|_{E_+}. \end{cases} \tag{78}$$

5 Generalized solvability

Consider a problem

$$\mathcal{L}_\varepsilon u = f, f \in E_+^-. \tag{79}$$

We will interpret its solutions in the following sense.

Definition 5.1. The function $u(x,t) \in E$ is called the solution of the problem (79) with the right-hand side $f \in E_+^-$ if there exists a sequence of functions $u_i(x,t) \in C_{BR}^\infty$, for which the following holds

$$\|u - u_i\|_E \to 0, \|\mathcal{L}u_i - f\|_{E_+^-} \to 0, i \to \infty.$$

Definition 5.2. The function $u(x,t) \in E$ is called the weak solution of the problem (79) with the right-hand side $f \in E_+^-$, if the equality

$$(\mathcal{L}_\varepsilon u, v)_{E_+^-, E_+} = (f, v)_{W_+^-, W_+}$$

holds for some arbitrary functions $v \in C_{BR^*}^\infty$.

The solutions of the adjoint problem are defined in a similar way.

Based on the estimations (78) and on [4], we state the theorems of generalized solvability.

Theorem 5.1. *Definitions 5.1 ad 5.2 are equivalent.*

Theorem 5.2. *For an arbitrary element $f \in W_+^-$ there exists a unique solution of (79) in the sense of 5.1 and 5.2.*

Theorem 5.3. *Let $u(x,t)$ be the solution of the problem (79), with the right-hand side $f \in W_+^-$ in the sense of the definitions 5.1 and 5.2. Then, the following estimation holds*

$$\|u\|_E \le c\|f\|_{W_+^-},$$

where the constant c does not depend on f.

Similar theorems can be stated for the adjoint problem.

6 Results

In this work we considered an integro-differential equation with a small time derivative coefficient. We used a method of a priori estimations in negative norms which helped us to prove well-posedness of the Dirichlet problem with initial and boundary conditions. Finally, we achieved the following results:

- We suggested a way to apply the method of a priori estimations in negative norms for integro-differential equations of Volterra type.

- The kernels of the integral operator are not supposed to be the same, which means that different second derivatives of the unknown function can be multiplied by different kernels. Moreover, we do not need the non-negative condition for the kernels.

- We proved the existence and uniqueness of the generalized solution and it's continuous dependence on the right-hand side of the equation.

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